# Periodic sequences modulo $m$

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**Abstract**

We give a few remarks on the periodic sequence $a_n = \binom{n}{x} \ (mod\ m)$ where $x, m, n \in \mathbb{N}$, which is periodic with minimal length of the period being

$$l(m,x) = \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i} x \rfloor + b_i} = m \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i} x \rfloor}$$

where $m = \prod_{i=1}^{k} p_i^{b_i}$. We also give a new proof of that result and prove certain interesting properties of $l(m,x)$ and derive a few other results.

## 1 Introduction and Preliminaries

The authors in [2] stated and proved the following

**Theorem 1.** *A natural number $p > 1$ is a prime if and only if $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by $p$ for every non-negative $n$, where $n > p + 1$ and the symbols have their usual meanings.*

The proof of Theorem 1 was completed by Laugier nd Saikia [1]. In this section we state without proof the following results which we shall be referring in the coming sections. The proofs can be found in [3].

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**Definition 2.** A sequence $(a_n)$ is said to be periodic modulo $m$ with period $k$ if there exists an integer $N > 0$ such that for all $n > N$

$$a_{n+k} = a_n \pmod{m}.$$

**Theorem 3.** *The sequence* $(a_n) = \binom{n}{x}$ *(mod m) is periodic, where* $x, m, n \in \mathbb{N}$.

**Theorem 4.** *For a natural number* $m = \prod_{i=1}^{k} p_i^{b_i}$, *the sequence* $a_n \equiv \binom{n}{m}$ *(mod m) has a period of minimal length,*

$$l(m) = \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i} m \rfloor + b_i}.$$

The following generalization of Theorem 1 was also proved in [1]

**Theorem 5.** *For* $k = 0$, *we set the convention that* $a_{(0)} = n = a_0 + a_1 p + \ldots + a_l p^l$ *and* $b_{(0)} = 0$. *For* $n = ap + b = a_{(k)} p^k + b_{(k)}$, *we have*

$$\binom{a_{(k)} p^k + b_{(k)}}{p^k} - \left\lfloor \frac{a_{(k)} p^k + b_{(k)}}{p^k} \right\rfloor \equiv 0 \pmod{p}$$

*with* $p$ *a prime,* $0 \leq b_{(k)} \leq p^k - 1$ *and* $k$ *a positive integer such that* $1 \leq k \leq l$.

*Here* $n = a_0 + a_1 p + \ldots + a_k p^k + a_{k+1} p^{k+1} + \ldots + a_l p^l$, *and we have for* $k \geq 1$

$$a_{(k)} = a_k + a_{k+1} p + \ldots + a_l p^{l-k}$$

*and*

$$b_{(k)} = a_0 + a_1 p + \ldots + a_{k-1} p^{k-1}.$$

*In particular, we have*

$$a = a_{(1)} = a_1 + a_2 p + \ldots + a_l p^{l-1}$$

*and*

$$b = b_{(0)} = a_0.$$

Notice that Theorem 5 is obviously true for $k = 0$. But the case $k = 0$ doesn't correspond really to a power of $p$ where $p$ is a prime.

We also fix the notation $[[1, i]]$ for the set $\{1, 2, \ldots, i\}$ throughout the paper.

**Definition 6.** We define $\mathrm{ord}_p(n)$ for $n \in \mathbb{N}$ to be the greatest exponent of $p$ with $p$ a prime in the decomposition of $n$ into prime factors,

$$\mathrm{ord}_p(n) = \max\left\{k \in \mathbb{N} \,:\, p^k | n\right\}.$$

## 2 Remarks on Theorem 3

The integer $n$ in Theorem 3 should be greater than $x$. Otherwise, the binomial coefficient $\binom{n}{x}$ is not defined. But, we can extend the definition of $\binom{n}{x}$ to integer $n$ such that $0 \leq n < x$ by setting $\binom{n}{x} = 0$ if $0 \leq n < x$. Nevertheless, notice that this extension is not necessary in order to prove this theorem about periodic sequences.

The case where $m = 0$ is not possible since the sequence $(\binom{n}{x})$ is not periodic modulo 0 or is not simply periodic. So, if $x = m$, $x$ should be non-zero.

If $x = 0$, then we have

$$a_n \equiv a_{n+1} \equiv \ldots \equiv a_{n+k} \equiv 1 \pmod{m}$$

for any integers $n$ and $k$. So, if $x = 0$, the sequence $(a_n)$ is periodic with minimal period equal to 1. We recall that if a sequence is periodic, a period of such a sequence is a non-zero integer.

In the following, we assume $x \geq 1$.

We give a proof of Theorem 3 with the help of the following two Lemmas.

**Lemma 7.** *For $n \geq x + 1$*

$$\sum_{i=x}^{n-1} \binom{i}{x} = \binom{n}{x+1}.$$

*Proof.* We prove this property by induction.

For $n = x + 1$, we have

$$\sum_{i=x}^{x} \binom{i}{x} = \binom{x}{x} = \binom{x+1}{x+1} = 1$$

Since

$$\sum_{i=x}^{n-1} \binom{i}{x} = \binom{n}{x+1}$$

So we have

$$\sum_{i=x}^{n} \binom{i}{x} = \sum_{i=x}^{n-1} \binom{i}{x} + \binom{n}{x} = \binom{n}{x+1} + \binom{n}{x} = \binom{n+1}{x+1}$$

where we used the Pascal's rule.

□

Let $k$ be the length of a period of sequence $a_n \equiv \binom{n}{x} \pmod{m}$, meaning $\binom{n+k}{x} \equiv \binom{n}{x} \pmod{m}$. Then we have

**Lemma 8.**

$$\sum_{j=x}^{x+mk-1} \binom{j}{x} \equiv 0 \pmod{m}.$$

*Proof.* Let

$$\sum_{j=x}^{x+k-1} \binom{j}{x} \equiv r \pmod{m}.$$

Moreover

$$\sum_{j=x}^{x+mk-1} \binom{j}{x} = \sum_{j=x}^{x+k-1} \binom{j}{x} + \sum_{j=x+k}^{x+2k-1} \binom{j}{x} + \ldots + \sum_{j=x+(m-1)k}^{x+mk-1} \binom{j}{x}$$

or equivalently, in a more compact way

$$\sum_{j=x}^{x+mk-1} \binom{j}{x} = \sum_{i=1}^{m} \sum_{j=x+(i-1)k}^{x+ik-1} \binom{j}{x}.$$

Performing the change of label $j \to l = j-(i-1)k$ in $\sum_{j=x+(i-1)k}^{x+ik-1} \binom{j}{x}$, we have

$$\sum_{j=x+(i-1)k}^{x+ik-1} \binom{j}{x} = \sum_{l=x}^{x+k-1} \binom{l+(i-1)k}{x} = \sum_{l=x}^{x+k-1} \binom{l}{x}$$

where we used the fact that: $\binom{l+(i-1)k}{x} = \binom{l}{x}$.

Since $l$ is a dummy running index, we can replace $l$ by $j$ and we obtain

$$\sum_{j=x+(i-1)k}^{x+ik-1} \binom{j}{x} = \sum_{j=x}^{x+k-1} \binom{j}{x}.$$

Therefore

$$\sum_{j=x}^{x+mk-1} \binom{j}{x} = \sum_{i=1}^{m} \sum_{j=x}^{x+k-1} \binom{j}{x} \equiv \sum_{i=1}^{m} r \pmod{m}.$$

Thus

$$\sum_{j=x}^{x+mk-1} \binom{j}{x} \equiv mr \equiv 0 \pmod{m}.$$

$\square$

We now have,

$$\sum_{j=x}^{n+mk-1} \binom{j}{x} = \sum_{j=x}^{x+mk-1} \binom{j}{x} + \sum_{j=x+mk}^{n+mk-1} \binom{j}{x} \equiv \sum_{j=x+mk}^{n+mk-1} \binom{j}{x} \pmod{m}.$$

Performing the change of label $j \to l = j-mk$ in $\sum_{j=x+mk}^{n+mk-1} \binom{j}{x}$, we have

$$\sum_{j=x+mk}^{n+mk-1} \binom{j}{x} = \sum_{l=x}^{n-1} \binom{l+mk}{x}.$$

Since $\binom{l+mk}{x} = \binom{l}{x}$ and since $l$ is a dummy running index, replacing $l$ by $j$ and using Lemma 7, we have

$$\sum_{j=x+mk}^{n+mk-1} \binom{j}{x} = \sum_{j=x}^{n-1} \binom{j+mk}{x} = \sum_{j=x}^{n-1} \binom{j}{x} = \binom{n}{x+1}$$

and we deduce that

$$\sum_{j=x+mk}^{n+mk-1} \binom{j}{x} \equiv \binom{n}{x+1} \pmod{m}$$

Using again Lemma 7, we have:

$$\sum_{j=x+mk}^{n+mk-1} \binom{j}{x} = \binom{n+mk}{x+1}.$$

From $\displaystyle\sum_{j=x+mk}^{n+mk-1} \binom{j}{x} \equiv \binom{n}{x+1} \pmod{m}$ and $\displaystyle\sum_{j=x+mk}^{n+mk-1} \binom{j}{x} = \binom{n+mk}{x+1}$, we get

$$\binom{n+mk}{x+1} \equiv \binom{n}{x+1} \pmod{m}.$$

Thus, since $a_n \equiv \binom{n}{x} \pmod{m}$, we have

$$a_{n+k} \equiv \binom{n+k}{x} \equiv \binom{n}{x} \equiv a_n \pmod{m}$$

We conclude that the sequence $(a_n)$ such that $a_n \equiv \binom{n}{x} \pmod{m}$ is periodic.

Thus we have outlined an alternate proof of Theorem 3. We now state and prove a generalization of Lemma 8.

**Lemma 9.**

$$\sum_{j=n}^{n+mk-1} \binom{j}{x} \equiv 0 \pmod{m}$$

*where it is understood that if $n$ is strictly less than $x$, then for $n \leq j < x$, $\binom{j}{x}$ cancels out.*

*Proof.* The proof of Lemma 9 follows from the proof of Lemma 8. Indeed, in the proof of Lemma 8, it suffices to replace sums like $\sum_{j=x}^{x+mk-1} \binom{j}{x}$, $\sum_{j=x+(i-1)k}^{x+ik-1} \binom{j}{x}$ with $i = 1, 2, \ldots, m$ by respectively $\sum_{j=n}^{n+mk-1} \binom{j}{x}$, $\sum_{j=n+(i-1)k}^{n+ik-1} \binom{j}{x}$. And, in order to proceed like the proof of Lemma 8, we can call $r_n$ the remainder when $\sum_{j=n}^{n+k-1} \binom{j}{x}$ is divided by $m$.

Notice that the change of label $j \to l = j - (i-1)k$ in $\displaystyle\sum_{j=n+(i-1)k}^{n+ik-1} \binom{j}{x}$ is performed like before in the proof of Lemma 8 giving us

$$\sum_{j=n+(i-1)k}^{n+ik-1} \binom{j}{x} = \sum_{l=n}^{n+k-1} \binom{l+(i-1)k}{x} = \sum_{l=n}^{n+k-1} \binom{l}{x}$$

where we used the fact that $\binom{l+(i-1)k}{x} = \binom{l}{x}$. $\square$

## 3 Remarks on Theorem 4

In [1], the authors mention without proof the following generalization of Theorem 4.

**Theorem 10.** *For a natural number* $m = \prod_{i=1}^{k} p_i^{b_i}$*, the sequence* $(a_n)$ *such that* $a_n \equiv \binom{n}{x} \pmod{m}$ *has a period of minimal length*

$$l(m, x) = \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i} x \rfloor + b_i} = m \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i} x \rfloor}.$$

The proof follows from the proof of Theorem 4 as given in [3]. An easy corollary mentioned in [1] is proved below

**Corollary 11.** *For* $m = \prod_{i=1}^{k} p_i^{b_i}$ *we have*

$$m^2 \leq l(m) \leq m^{k+1}.$$

*Proof.* We have

$$\lfloor \log_{p_i}(m) \rfloor = \lfloor \log_{p_i} \left( \prod_{j=1}^{k} p_j^{b_j} \right) \rfloor = b_i + \lfloor \sum_{j \in [[1,k]] - \{i\}} b_j \log_{p_i}(p_j) \rfloor \geq b_i.$$

This implies

$$l(m) = m \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i}(m) \rfloor} \geq m \prod_{i=1}^{k} p_i^{b_i}.$$

So $l(m) \geq m^2$.

Notice that

$$l(m) = m \prod_{i=1}^{k} p_i^{b_i + \lfloor \sum_{j \in [[1,k]] - \{i\}} b_j \log_{p_i}(p_j) \rfloor} = m^2 \prod_{i=1}^{k} p_i^{\lfloor \sum_{j \in [[1,k]] - \{i\}} b_j \log_{p_i}(p_j) \rfloor}.$$

So, we verify that $l(m)$ is divisible by $m^2$. Moreover, we have

$$\lfloor \log_{p_i}(m) \rfloor \leq \log_{p_i}(m).$$

It now follows

$$\lfloor \log_{p_i}(m) \rfloor \leq b_i + \log_{p_i} \left( \prod_{j \in [[1,k]] - \{i\}} p_j^{b_j} \right)$$

$$l(m) = m \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i}(m) \rfloor} \leq m \left( \prod_{i=1}^{k} p_i^{b_i} \right) \prod_{i=1}^{k} \left( \prod_{j \in [[1,k]] - \{i\}} p_j^{b_j} \right)$$

That is

$$l(m) \leq m^2 \left( \prod_{i=1}^{k} p_i^{b_i} \right)^{k-1} = m^2 \times m^{k-1}.$$

So $l(m) \leq m^{k+1}$.

□

*Remark* 12. Here $k \leq m - \varphi(m)$ where $\varphi$ is the Euler totient function.

**Definition 13** (Minimal Period of a periodic sequence)**.** The period of minimal length of a periodic sequence $(a_n)$ such that $a_n \equiv \binom{n}{x} \pmod{m}$ with $x \in \mathbb{N}$ and $m \in \mathbb{N}$, is the minimal non-zero natural number $\ell(m, x)$ such that for all positive integer $n$ we have

$$\binom{n + \ell(m, x)}{x} \equiv \binom{n}{x} \pmod{m}$$

where it is understood that

$$\binom{n}{x} = \begin{cases} 0, & \text{if} \quad 0 \leq n < x \\ \frac{n!}{x!(n-x)!}, & \text{if} \quad n \geq x. \end{cases}$$

*Remark* 14. If $x = 0$, then $\ell(m, x = 0) = 1$ with $m \in \mathbb{N}$.

From Definition 13

$$\binom{\ell(m, x)}{x} \equiv \binom{\ell(m, x) + 1}{x} \equiv \cdots \equiv \binom{\ell(m, x) + x - 1}{x} \equiv 0 \pmod{m}.$$

If $x > 0$ ($x \in \mathbb{N}$), since any number is divisible by 1, we have

$$\binom{x}{x} \equiv \binom{x+1}{x} \equiv \cdots \equiv \binom{2x-1}{x} \equiv 0 \pmod{1}.$$

Regarding the definition of $\ell(m, x)$, since $x$ is the least non-zero natural number which verifies this property, we can set ($x \in \mathbb{N}$) $\ell(1, x) = x$.

The minimal period $\ell(m)$ of a sequence $(a_n)$ such that $a_n \equiv \binom{n}{m} \pmod{m}$ with $m \in \mathbb{N}$ (see Theorem 4) is given by $\ell(m) = \ell(m, m)$.

Before we mention a few results we recall that $\log_a x = \frac{\ln x}{\ln a}$ and $\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.

We can notice that for $x > 0$ we have

$$\log_p(x + 1) = \log_p(x) + \log_p\left(1 + \frac{1}{x}\right).$$

The series expansion of $\log_p\left(1 + \frac{1}{x}\right)$ near $+\infty$ up to order 1 in the variable $1/x$ is given by

$$\log_p\left(1 + \frac{1}{x}\right) = \frac{1}{x \ln p} + o\left(\frac{1}{x}\right).$$

Therefore, we have

$$\log_p(x + 1) = \log_p(x) + \frac{1}{x \ln p} + o\left(\frac{1}{x}\right).$$

**Theorem 15.**

$$\lfloor \log_p(x + 1) \rfloor = \lfloor \log_p(x) + \frac{1}{x \ln p} \rfloor = \begin{cases} \lfloor \log_p(x) \rfloor, & \text{if} \quad x \neq p^c - 1; \\ \lfloor \log_p(x) \rfloor + 1, & \text{if} \quad x = p^c - 1, \end{cases}$$

*with* $c \in \mathbb{N}$.

*Proof.* `Case I.`

Let us take $x = p^c - 1$. Then

$$\log_p(x+1) = \log_p(p^c) = c \log_p(p) = c,$$

and so

$$\lfloor \log_p(x+1) \rfloor = \lfloor c \rfloor = c.$$

Thus, $\log_p(x+1) = \lfloor \log_p(x+1) \rfloor = c$. When $c = 1$ and $p = 2$, the relation $\lfloor \log_p(x+1) \rfloor = \lfloor \log_p(x) + \frac{1}{x \ln p} \rfloor = \lfloor \log_p(x) \rfloor + 1$ is true.

In the following, we assume that one of the conditions $c > 1$ and $p > 2$ is true; so $x > 1$. We have:

$$\log_p(p^c - 1) = c + \log_p\left(1 - \frac{1}{p^c}\right). \tag{1}$$

The series expansion of $\log_p\left(1 - \frac{1}{p^c}\right)$ near $+\infty$ is given by

$$\log_p\left(1 - \frac{1}{p^c}\right) = -\frac{1}{\ln p}\sum_{k=1}^{+\infty} \frac{1}{k \cdot p^{kc}}.$$

We have

$$\left|\log_p\left(1 - \frac{1}{p^c}\right)\right| < \frac{1}{\ln p}\sum_{k=1}^{+\infty} \frac{1}{p^{kc}}.$$

So for $c > 1$ or/and $p > 2$ we get

$$\left|\log_p\left(1 - \frac{1}{p^c}\right)\right| < \frac{1}{(p^c - 1)\ln p} < 1. \tag{2}$$

From (1) and (2) we get an $\epsilon_p \in (0, 1)$ such that

$$\log_p(x) = c - \epsilon_p. \tag{3}$$

Notice that

$$0 < \epsilon_p < \frac{1}{(p^c - 1)\ln p} < 1. \tag{4}$$

It implies that for $x = p^c - 1$ with $c > 1$ or/and with $p > 2$ we get

$$\lfloor \log_p(x) \rfloor = c - 1 = \lfloor \log_p(x+1) \rfloor - 1.$$

Moreover from (3) and (4) we have

$$c < \log_p(x) + \frac{1}{x \ln p} < c + 1 - \epsilon_p.$$

Since $0 < 1 - \epsilon_p < 1$ for $\epsilon \in (0, 1)$ and $\lfloor \log_2(2) \rfloor = \lfloor \log_2(1) + \frac{1}{\ln 2} \rfloor = \lfloor \log_2(1) \rfloor + 1 = 1$, thus for $x = p^c - 1$ with $c \geq 1$ we have

$$\lfloor \log_p(x+1) \rfloor = \lfloor \log_p(x) + \frac{1}{x \ln p} \rfloor = \lfloor \log_p(x) \rfloor + 1 = c.$$

`Case II.`

If $x+1$ is not a power of a prime, for given $x$ and for $p$ a prime, there exists $c \geq 1$ such that $p^c \leq x \leq p^{c+1}-2$. We can take $x = p^c - 1 + y$ with $1 \leq y \leq p^{c+1} - p^c - 1$. Then,

$$\log_p(x+1) = \log_p(p^c + y) = c + \log_p\left(1 + \frac{y}{p^c}\right). \tag{5}$$

Since $\frac{1}{p^c} \leq \frac{y}{p^c} \leq p - 1 - \frac{1}{p^c}$, we have

$$c + \log_p\left(1 + \frac{1}{p^c}\right) \leq \log_p(x+1) \leq c + \log_p\left(p - \frac{1}{p^c}\right).$$

So

$$c + \log_p\left(1 + \frac{1}{p^c}\right) \leq \log_p(x+1) \leq c + 1 + \log_p\left(1 - \frac{1}{p^{c+1}}\right).$$

We can find an $\epsilon'_p \in (0,1)$ such that

$$c + \epsilon'_p \leq \log_p(x+1) \leq c + 1 - \epsilon'_p. \tag{6}$$

So

$$\lfloor \log_p(x+1) \rfloor = c.$$

Notice that we must have

$$c + \epsilon'_p \leq c + \log_p\left(1 + \frac{1}{p^c}\right) \leq \log_p(x+1) \leq c + 1 + \log_p\left(1 - \frac{1}{p^{c+1}}\right) \leq c + 1 - \epsilon'_p.$$

So

$$\log_p\left(1 + \frac{1}{p^c}\right) \leq \epsilon'_p \leq \left|\log_p\left(1 - \frac{1}{p^{c+1}}\right)\right|. \tag{7}$$

Moreover, we have

$$\log_p(x) = \log_p(p^c - 1 + y) = c + \log_p\left(1 - \frac{1}{p^c} + \frac{y}{p^c}\right).$$

Since $\frac{1}{p^c} \leq \frac{y}{p^c} \leq p - 1 - \frac{1}{p^c}$, we have

$$c \leq \log_p(x) \leq c + 1 + \log_p\left(1 - \frac{2}{p^{c+1}}\right).$$

By standard analysis, it can be shown that for $t \geq p$,

$$0 < \left|\log_p\left(1 - \frac{2}{t^{c+1}}\right)\right| < 1.$$

So, taking $t = p$, it implies

$$0 < \left|\log_p\left(1 - \frac{2}{p^{c+1}}\right)\right| < 1.$$

Thus there exists an $\eta_p \in (0,1)$ such that for $p^c \leq x \leq p^{c+1} - 2$,

$$c \leq \log_p(x) \leq c + 1 + \eta_p.$$

Therefore for $p^c \leq x \leq p^{c+1} - 2$ we have

$$c \leq \lfloor \log_p(x) \rfloor \leq c + 1,$$

and

$$\lfloor \log_p(x) \rfloor + 1 \geq c + 1.$$

Consequently from (5) and (7)

$$\lfloor \log_p(x+1) \rfloor = c$$

and we have for $p^c \leq x \leq p^{c+1} - 2$

$$\lfloor \log_p(x) \rfloor + 1 > \lfloor \log_p(x+1) \rfloor.$$

We now show $\lfloor \log_p(x) + \frac{1}{x \ln p} \rfloor = c$.

By standard analysis, it can be shown that for $t \geq p^c$ and $p$ a prime,

$$\left| \log_p(t) + \frac{1}{t \ln p} - \log_p(t+1) \right| < \frac{1}{p^c \ln p} - \log_p\left(1 + \frac{1}{p^c}\right).$$

Taking $t = x$ with $p^c \leq x \leq p^{c+1} - 2$, we get

$$\log_p(x+1) - \frac{1}{p^c \ln p} + \log_p\left(1 + \frac{1}{p^c}\right) < \log_p(x) + \frac{1}{x \ln p} < \log_p(x+1) + \frac{1}{p^c \ln p} - \log_p\left(1 + \frac{1}{p^c}\right).$$

From (6) we have

$$c + \epsilon'_p - \frac{1}{p^c \ln p} + \log_p\left(1 + \frac{1}{p^c}\right) < \log_p(x) + \frac{1}{x \ln p} < c + 1 - \epsilon'_p + \frac{1}{p^c \ln p} - \log_p\left(1 + \frac{1}{p^c}\right).$$

From (7) we get

$$c + 2\log_p\left(1 + \frac{1}{p^c}\right) - \frac{1}{p^c \ln p} < \log_p(x) + \frac{1}{x \ln p} < c + 1 + \frac{1}{p^c \ln p} - 2\log_p\left(1 + \frac{1}{p^c}\right).$$

Again by standard analysis, it can be shown for $t \geq 1$,

$$0 < 2\log_p(t+1) - 2\log_p(t) - \frac{1}{t \ln p} < 1.$$

So taking $t = p^c$, we have

$$0 < 2\log_p\left(1 + \frac{1}{p^c}\right) - \frac{1}{p^c \ln p} < 1.$$

Thus, we can find $\epsilon_p'' \in (0,1)$ such that

$$c + \epsilon_p'' < \log_p(x) + \frac{1}{x \ln p} < c + 1 - \epsilon_p''.$$

hence we get for $p^c \le x \le p^{c+1} - 2$

$$\lfloor \log_p(x) + \frac{1}{x \ln p} \rfloor = \lfloor \log_p(x+1) \rfloor = c < \lfloor \log_p(x) \rfloor + 1.$$

This completes the proof. $\square$

We now have the following

**Corollary 16.**

$$\ell(m, x+1) = \begin{cases} \ell(m,x), & \text{if} \quad x \neq p^c - 1 \quad \text{and} \quad p|m; \\ p\,\ell(m,x), & \text{if} \quad x = p^c - 1 \quad \text{and} \quad p|m, \end{cases}$$

*with* $x, m \in \mathbb{N}$.

The proof of the above corollary comes from Definition 13 and Theorem 15.
From Lemma 7

$$\sum_{j=x}^{x+k-1} \binom{j}{x} = \binom{x+k}{x+1}.$$

Notice that the formula is valid since $k = \ell(m,x)$ is an integer which is greater than 1. So, the binomial coefficient $\binom{x+k}{x+1}$ is well defined for $x \in \mathbb{N}$. Nevertheless, it was remarked in [3] that we can extend possibly the definition of $\binom{n}{x}$ (where it is implied that $0 \le x \le n$) to negative $n$.

Below we discuss a few general results and give a few general comments.

Using Pascal's rule, we can observe that

$$\binom{x+k}{x+1} + \binom{x+k}{x} = \binom{x+k+1}{x+1}.$$

Since $\binom{x+k}{x} \equiv \binom{x}{x} \equiv 1 \pmod{m}$, we obtain

$$\binom{x+k}{x+1} + 1 \equiv \binom{x+k+1}{x+1} \pmod{m}. \tag{8}$$

If $x \neq p^c - 1$ and $p|m$, then from the corollary above, we have $k = \ell(m,x) = \ell(m,x+1)$. So

$$\binom{x+k+1}{x+1} \equiv \binom{x+1}{x+1} \equiv 1 \pmod{m},$$

and hence

$$\binom{x+\ell(m,x)}{x+1} \equiv 0 \pmod{m}.$$

If $x = p^c - 1$ and $p|m$, then from the corollary above, we have $pk = p\,\ell(m,x) = \ell(m, x+1)$. Now from Theorem 5 we have for $x = p^c - 1$ and $m = p$ a prime with $c \in \mathbb{N}$,

$$\binom{x+k+1}{x+1} = \binom{p^c + \ell(p, p^c-1)}{p^c} \equiv \left\lfloor \frac{p^c + \ell(p, p^c-1)}{p^c} \right\rfloor \equiv \left\lfloor \frac{\ell(p, p^c-1)}{p^c} \right\rfloor + 1 \pmod p. \tag{9}$$

From (8) and (9) with $x = p^c - 1$, $k = \ell(m, x)$ and $m = p$ a prime we have

$$\binom{p^c - 1 + \ell(p, p^c-1)}{p^c} \equiv \left\lfloor \frac{\ell(p, p^c-1)}{p^c} \right\rfloor \pmod p.$$

We have $\ell(p, p^c) = p^{c+1} = p\,\ell(p, p^c - 1)$, so it follows that $\ell(p, p^c-1) = p^c$ and hence $\lfloor \frac{\ell(p,p^c-1)}{p^c} \rfloor = 1$ for $c \in \mathbb{N}$. Thus

$$\binom{2p^c - 1}{p^c} \equiv 1 \pmod p.$$

In general, if $x = p^c - 1$ and $p|m$, then since $\lfloor \log_p(p^c) \rfloor = c$, and from Corollary 16 we have

$$\ell(m, p^c - 1) = \frac{\ell(m, p^c)}{p} = mp^{c-1} \prod_{i \in [[1,k]] \,|\, p_i \neq p} p_i^{\lfloor \log_{p_i}(p^c) \rfloor}.$$

If $b_i = \mathrm{ord}_{p_i}(m) = \lfloor \log_{p_i}(p^c) \rfloor$ for $i \in [[1,k]] \,|\, p_i \neq p$ and $b = \mathrm{ord}_p(m)$, we write $m = m_c$ and we have

$$\ell(m_c, p^c - 1) = \frac{\ell(m_c, p^c)}{p} = m_c\, p^{c-1} \prod_{i \in [[1,k]] \,|\, p_i \neq p} p_i^{b_i} = m_c\, p^{c-1} \times \frac{m_c}{p^b}.$$

So, we deduce that

$$\ell(m_c, p^c - 1) = m_c^2\, p^{c-b-1} = \frac{m_c^2}{p^{b+1}} p^c,$$

and

$$\ell(m, p^c) = m_c^2\, p^{c-b}.$$

In particular, when $m_c = p^b$ we have $\ell(m_c = p^b, p^c - 1) = p^{b+c-1}$. And, we get $\ell(m_c = p, p^c - 1) = p^c$. If $c \geq \mathrm{ord}_p(m_c) + 1$, then $\ell(m_c, p^c - 1)$ is divisible by $m^2$. If $b = c$, then $\ell(m_c = p^c, p^c - 1) = \frac{m_c^2}{p}$.

Now from (9) we have

$$\binom{x+k+1}{x+1} = \binom{p^c + \ell(m_c, p^c-1)}{p^c} \equiv \left\lfloor \frac{p^c + \ell(m_c, p^c-1)}{p^c} \right\rfloor \equiv \left\lfloor \frac{m_c^2}{p^{b+1}} \right\rfloor + 1 \pmod p.$$

From (8) with $x = p^c - 1$, $k = \ell(m, x)$, and also from the fact that $d \equiv e \pmod m$ and $p|m$ implies that $d \equiv e \pmod p$ (the converse is not always true), we have for $p|m_c$,

$$\binom{p^c - 1 + m_c^2\, p^{c-b-1}}{p^c} \equiv \left\lfloor \frac{m_c^2}{p^{b+1}} \right\rfloor \pmod p.$$

In the proof of Theorem 4, the authors in [3] first proved that a period of a sequence $(a_n)$ such that $a_n \equiv \binom{n}{m} \pmod{m}$ with $m = \prod_{i=1}^{k} p_i^{b_i}$ (with at least one non-zero $b_i$), should be a multiple of the number $\ell(m) = m \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i}(m) \rfloor}$. Afterwards, it is proved that $\ell(m)$ represents really the minimal period of such a sequence namely for every natural number $n$,

$$\binom{n + \ell(m)}{m} \equiv \binom{n}{m} \pmod{m}.$$

For that, the authors notice that it suffices to prove

$$\frac{\prod_{i=0}^{m-1}(n-i)}{\prod_{j=1}^{k} p_j^{\vartheta_{p_j}(m)}} \equiv \frac{\prod_{i=0}^{m-1}(n+\ell(m)-i)}{\prod_{j=1}^{k} p_j^{\vartheta_{p_j}(m)}} \pmod{m},$$

where $\vartheta_{p_j}(m)$ is the $p_j$-adic ordinal of $m!$ defined as

$$\vartheta_{p_j}(m) = \mathrm{ord}_{p_j}(m!) = \sum_{l \geq 1} \left\lfloor \frac{m}{p_j^l} \right\rfloor = \sum_{l=1}^{\lfloor \log_{p_j}(m) \rfloor} \left\lfloor \frac{m}{p_j^l} \right\rfloor. \tag{10}$$

Thus to prove Theorem 4 it is sufficient to show

$$\frac{\prod_{i=1}^{m}(n-i+1)}{\prod_{j=1}^{k} p_j^{\vartheta_{p_j}(m)}} \equiv \frac{\prod_{i=1}^{m}(n+\ell(m)-i+1)}{\prod_{j=1}^{k} p_j^{\vartheta_{p_j}(m)}} \pmod{m}.$$

Then, the authors observe that among the numbers $n, n-1, \ldots, n-m+1$, there are at least $\lfloor \frac{m}{p^l} \rfloor$ that are divisible by $p^l$ for every positive integer $l$ and any prime $p$ which appears in the prime factorization of $m$. In particular, if $p$ divides $m$, we can notice that among the numbers $n, n-1, \ldots, n-m+1$ (which represents $m$ consecutive numbers), there are exactly $\lfloor \frac{m}{p} \rfloor = \frac{m}{p}$ that are divisible by $p$ for any prime $p$ which appears in the prime factorization of $m$.

In the following, we define natural numbers $c_j(i)$ with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k$ by

$$\vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i)$$

such that the $c_j(i)$'s are functions of $\mathrm{ord}_{p_j}(n-i+1)$ namely $c_j(i) = (\mathrm{ord}_{p_j}(n-i+1))$ and $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k$. Also $c_j(i) = 0$ if $\mathrm{ord}_{p_j}(n-i+1) = 0$. (In general, the converse is not always true. Indeed, it may be possible that $c_j(i) = 0$ for some $n-i+1$ which have non-zero $p_j$-adic ordinal with $j = 1, 2, \ldots, k$). Thus, each number $c_j(i)$ is associated to each number $n-i+1$ in the sense that the number $c_j(i)$ depends on $\mathrm{ord}_{p_j}(n-i+1)$.

Thus we can now state and prove

**Theorem 17.** *If* $\max\left\{ c_j(i) \,\middle|\, \vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i) \right\} \leq \lfloor \log_{p_j}(m) \rfloor$ *then* $\vartheta_{p_j}(m) \leq \lfloor \frac{m}{p_j} \rfloor \lfloor \log_{p_j}(m) \rfloor$. *(In general, the converse is not always true.) Therefore, a necessary but not sufficient condition in order to satisfy the inequality* $\vartheta_{p_j}(m) \leq \lfloor \frac{m}{p_j} \rfloor \lfloor \log_{p_j}(m) \rfloor$, *is*

$$c_j(i) \leq \lfloor \log_{p_j}(m) \rfloor, \ \ \forall\, i \in [[1, m]]$$

*with* $j = 1, 2, \ldots, k$.

*Proof.* We have

$$\frac{\prod_{i=1}^{m}(n-i+1)}{\prod_{j=1}^{k} p_j^{\vartheta_{p_j}(m)}} = \frac{\prod_{i=1}^{m}(n-i+1)}{\prod_{j=1}^{k} p^{\sum_{i=1}^{m} c_j(i)}} = \frac{\prod_{i=1}^{m}(n-i+1)}{\prod_{j=1}^{k}\prod_{i=1}^{m} p^{c_j(i)}} = \frac{\prod_{i=1}^{m}(n-i+1)}{\prod_{i=1}^{m}\prod_{j=1}^{k} p^{c_j(i)}}.$$

Thus we can have

$$\frac{\prod_{i=1}^{m}(n-i+1)}{\prod_{j=1}^{k} p_j^{\vartheta_{p_j}(m)}} = \prod_{i=1}^{m} \frac{(n-i+1)}{\prod_{j=1}^{k} p^{c_j(i)}}.$$

Now since $\lfloor \frac{m}{p^l} \rfloor \leq \lfloor \frac{m}{p} \rfloor$ with $l \geq 1$, we have

$$\vartheta_{p_j}(m) = \sum_{l=1}^{\lfloor \log_{p_j}(m) \rfloor} \left\lfloor \frac{m}{p_j^l} \right\rfloor \leq \sum_{l=1}^{\lfloor \log_{p_j}(m) \rfloor} \left\lfloor \frac{m}{p_j} \right\rfloor.$$

Consequently

$$\vartheta_{p_j}(m) \leq \left\lfloor \frac{m}{p_j} \right\rfloor \lfloor \log_{p_j}(m) \rfloor,$$

and so

$$\sum_{i=1}^{m} c_j(i) \leq \left\lfloor \frac{m}{p_j} \right\rfloor \lfloor \log_{p_j}(m) \rfloor.$$

Again since among the numbers $n, n-1, \ldots, n-m+1$, there are exactly $\lfloor \frac{m}{p_j} \rfloor$ numbers which have non-zero $p_j$-adic ordinal, we have

$$\sum_{i=1}^{m} c_j(i) \leq \left\lfloor \frac{m}{p_j} \right\rfloor \max\left\{ c_j(i) \;\middle|\; \vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i) \right\}.$$

So, a necessary but not sufficient condition in order to have $\vartheta_{p_j}(m) \leq \lfloor \frac{m}{p_j} \rfloor \lfloor \log_{p_j}(m) \rfloor$ is

$$\max\left\{ c_j(i) \;\middle|\; \vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i) \right\} \leq \lfloor \log_{p_j}(m) \rfloor.$$

□

We can notice that this choice is not unique. But, we can observe that all the choices for the $c_j(i)$'s are equivalent in the sense that the equality $\vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i)$ should hold, meaning that we can come back to a decomposition of the value of $\vartheta_{p_j}(m)$ into sum of positive numbers like the $c_j(i)$'s for which $c_j(i) \leq \lfloor \log_{p_j}(m) \rfloor$ with $i = 1, 2, \ldots, m$. It turns out to be that this choice is suitable in order to prove that $\ell(m)$ is the minimal period of sequences $(a_n)$ such that $a_n \equiv \binom{n}{m} \pmod{m}$ with $m = \prod_{i=1}^{k} p_i^{b_i}$ (with at least one non-zero $b_i$).

*Remark* 18. We have obviously

$$\vartheta_{p_j}(m) = \sum_{l=1}^{\lfloor \log_{p_j}(m) \rfloor} \left\lfloor \frac{m}{p_j^l} \right\rfloor \geq \left\lfloor \frac{m}{p_j} \right\rfloor$$

and so

$$\sum_{i=1}^{m} c_j(i) \geq \left\lfloor \frac{m}{p_j} \right\rfloor.$$

Thus

$$\left\lfloor \frac{m}{p_j} \right\rfloor \max \left\{ c_j(i) \;\middle|\; \vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i) \right\} \geq \sum_{i=1}^{m} c_j(i)$$

and hence

$$\max \left\{ c_j(i) \;\middle|\; \vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i) \right\} \geq 1.$$

The above discussion gives us a motivation to study the coefficients $c_j(i)$'s. We hope to address a few issues related to them and establish some interesting results in a forthcoming paper.

## References


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